

Monotonic Mappings Invariant Linearisation of Finite Posets

Nicolás Madrid Labrador¹ and Umberto Straccia²

¹ Dept. Matemática Aplicada, Universidad de Málaga, Spain

² Istituto di Scienza e Tecnologie dell'Informazione (ISTI - CNR), Pisa, Italy

Abstract. In this paper we describe a novel procedure to build a linear order from an arbitrary poset which (i) preserves the original ordering and (ii) allows to extend monotonic and antitonic mappings defined over the original poset to monotonic and antitonic mappings over the new linear poset.

1 Introduction

Sorting the entry data as an optimal queue in order to optimise resources is a problem that is resolved in some specific cases only, as for example, when the algorithm deals with monotonic mappings and the data has a linear poset structure, as e.g. it arises in the context of top-k retrieval [4, 6], which originated the problem addressed here.³

Nevertheless the scoring data may not be structured as a linear lattice in general. Such cases arises, e.g., when the score is an interval $[a, b]$, e.g. indicating the probability of an answer being true. In such cases, one still wants to rank the answers in order of “highest” probability, but now the order is a partial order only.

A possible solution to this drawback is by redefining a linear ranking in the data. For example [5] resolves this drawback by defining a linear ordering through a “*ranking*” operator, which transforms each data-tuple into a value $\alpha \in \mathbb{R}$. Another possibility is to use *linear extensions* such as described in [1, 2, 7, 8] in which a poset (L, \leq) is extended to a linear poset (L, \leq') .

However none of these *linearisations methods* allow to handle appropriately monotonic mappings, in the sense that monotonic and antitonic mappings defined over the original poset can be extended to monotonic and antitonic mappings over the new linear poset (see Section 5).

The contribution of this paper is the following. We will define a new linear poset (\mathcal{L}, \preceq) from an arbitrary poset (L, \leq) by satisfying three convenient conditions. Firstly, each element in L is assigned to an element in \mathcal{L} ; i.e there is a mapping $p: L \rightarrow \mathcal{L}$. Secondly, the new linear lattice (\mathcal{L}, \preceq) will be conservative

³ In top-k retrieval one usually is interested in ranking and selecting the top-k answers to a query according to the answer’s score, where the score has been computed using some monotone function such as $f: \mathbb{R}^n \rightarrow \mathbb{R}$. In this case, f monotone iff $f(\mathbf{x}) \leq f(\mathbf{y})$ whenever $\mathbf{x} \leq \mathbf{y}$, where $\mathbf{x} \leq \mathbf{y}$ holds when $x_i \leq y_i$ for all i .

with respect to the original ordering, that is if $x < y$ in (L, \leq) then $p(x) \prec p(y)$ in (\mathcal{L}, \preceq) . Thirdly, if (L, \leq) is a linear poset then $(L, \leq) \equiv (\mathcal{L}, \preceq)$.

This new poset (\mathcal{L}, \preceq) will be called the linearisation of (L, \leq) . Given a linearisation, the elements in the poset can be ranked linearly by using the value assigned in the linearisation. The improvement with respect to other linear extensions is that we will be able to extend any monotonic mapping defined over (L, \leq) to a monotonic mapping defined over (\mathcal{L}, \preceq) . Hence, the rank (or ordering) provided by the linearisation respects the monotonicity of all monotonic mappings defined over (L, \leq) .

In the following, we proceed as follows. In Section 2 we introduce the notation and basic definitions needed to define our linearisation procedure. In Section 3 we define two linearisation procedures. In Section 4 we show that monotonic and antitonic mappings can be extended to the linearisation by preserving monotonicity and antitonicity over these linearisations. Finally in Section 5 we briefly compare the linearisation described in this paper with the linear extensions described in the literature and conclude.

2 Preliminaries

Let L be a set. A binary relation \leq determines an order relation in L if the following properties hold for all $x, y, z \in L$: $x \leq x$ for all $x \in L$ (*reflexivity*), if $x \leq y$ and $y \leq x$ then $x = y$ (*antisymmetry*) and if $x \leq y$ and $y \leq z$ then $x \leq z$ (*transitivity*). Moreover, if the order relation also satisfies the property: for all $x, y \in L$ either $x \leq y$ or $y \leq x$ (*linearity*) we say that \leq determines a *linear ordering* in L ; otherwise, \leq determines a *partial ordering* in L . A *poset* is pair (L, \leq) such that \leq is an order relation in L . A *chain* in a poset (L, \leq) is a subset $\{x_0, x_1, \dots, x_i, \dots\}$ of L such that for all $i \in \mathbb{N}$: $x_i \neq x_{i+1}$, $x_i \leq x_{i+1}$. Usually, the chain $\{x_0, x_1, \dots, x_i, \dots\}$ is denoted as $x_0 < x_1 < \dots < x_i < \dots$. The *cardinality* of a chain C is usually called the *length* of C . A chain C is called *maximal* if there is no chain C' such that $C \subset C'$. Given a poset (L, \leq) we extend \leq to L^n point wise, i.e. for $\mathbf{x}, \mathbf{y} \in L^n$, $\mathbf{x} \leq \mathbf{y}$ iff $x_i \leq y_i$, for all i . Now, let (L, \leq_L) and (M, \leq_M) be two posets. A mapping $f: L^n \rightarrow M^m$, where $f(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$ is called *monotonic* (resp. *antitonic*) if $\mathbf{x} \leq_L \mathbf{y}$ implies $f(\mathbf{x}) \leq_M f(\mathbf{y})$ (resp. $f(\mathbf{x}) \geq_M f(\mathbf{y})$).

Let \mathbb{X} be a set of subsets of a poset (L, \leq) . An order relation \sqsubseteq defined over \mathbb{X} determines a:

- *H-ordering* if $X \sqsubseteq Y$ imply for all $x \in X$ there is $y \in Y$ such that $x \leq y$ (Hoare order).
- *S-ordering* if $X \sqsubseteq Y$ imply for all $y \in Y$ there is $x \in X$ such that $x \leq y$ (Smyth order).
- *EM-ordering* if \sqsubseteq determines a H-ordering and a S-ordering (Egli-Milner order).

3 The linearisation of a finite poset

The aim of this section is to define, from a *finite* poset (L, \leq) , a linear poset (\mathcal{L}, \preceq) satisfying the properties described in the introduction; namely:

- If (L, \leq) is a linear poset then $(L, \leq) = (\mathcal{L}, \preceq)$
- There is a map $p : L \rightarrow \mathcal{L}$ such that $\forall x, y \in L$ if $x < y$ then $p(x) < p(y)$.

To start with, we begin by defining a set of disjoint sets in order to define later an equivalence relation over L . We proceed inductively as follows (recall that L is finite):

$$\begin{aligned}
 L_0 &= \{x \in L \mid \nexists y \in L \text{ such that } x < y\} \\
 L_1 &= \{x \in L \mid x \notin L_0, \nexists y \in L \setminus L_0 \text{ such that } x < y\} \\
 &\vdots \\
 L_i &= \{x \in L \mid x \notin L_0 \cup \dots \cup L_{i-1}, \\
 &\quad \nexists y \in L \setminus (L_0 \cup \dots \cup L_{i-1}) \text{ such that } x < y\} \\
 &\vdots
 \end{aligned}$$

Each L_i is called the i -level of L . The underlying idea under the levels is to take the values of L in decreasing order from the top elements. The following lemma can be shown:

Lemma 1. *Let (L, \leq) be a poset.*

1. *If $x \in L$ belongs to the level L_k then $x \notin L_i$ for all $i \neq k$.*
2. *If $x < y$, $x \in L_i$ and $y \in L_j$ then $i > j$.*
3. *If x and y belong to the same level then x and y are incomparable.*

The levels also satisfy the following property:

Proposition 1. *Let (L, \leq) be a finite poset. If $L_0 \cup \dots \cup L_{i-1} \neq L$ then $L_i \neq \emptyset$.*

Proof. Suppose $L_0 \cup \dots \cup L_{i-1} \neq L$, then there is an element $x_0 \in L \setminus (L_0 \cup \dots \cup L_{i-1})$. If $x_0 \notin L_i$ then there exists $x_1 \in L \setminus (L_0 \cup \dots \cup L_{i-1})$ such that $x_0 < x_1$. If $x_1 \notin L_i$ then there exists $x_2 \in L \setminus (L_0 \cup \dots \cup L_{i-1})$ such that $x_0 < x_1 < x_2$. This construction cannot be infinite since L is finite and therefore there exists $k \in \mathbb{N}$ such that $x_k \in L_i$. \square

Note that Proposition 1 does not hold if the poset is not finite. For example, $[0, 1]$ with the usual order does not satisfy Proposition 1, since $L_1 = \emptyset$ and $L_0 = \{1\} \neq [0, 1]$.

Proposition 2. *If $L_i = \emptyset$ then $L_{i+1} = \emptyset$ as well.*

Proof. As $L_i = \emptyset$ then:

$$\begin{aligned} L_{i+1} &= \{x \in L \mid x \notin L_0 \cup \dots \cup L_i, \nexists y \in L \setminus (L_0 \cup \dots \cup L_i) \text{ such that } x < y\} \\ &= \{x \in L \mid x \notin L_0 \cup \dots \cup L_{i-1}, \nexists y \in L \setminus (L_0 \cup \dots \cup L_{i-1}) \text{ such that } x < y\} \\ &= L_i . \end{aligned}$$

□

Corollary 1. *Let (L, \leq) be a finite poset. Each $x \in L$ belongs to one, and only one, level L_i of L .*

Corollary 1 shows us that we can define an equivalence relation over a finite poset (L, \leq) by identifying the elements which belong to the same level. That is, let x and y be two elements in L , we say that $x \sim y$ if and only if x belongs to the same level of y . In other words, $x \sim y$ if and only if there exists $i \in \mathbb{N}$ such that $x, y \in L_i$. It is easy to prove that the relation \sim is actually an equivalence relation since each element $x \in L$ belongs to one, and only one, level of L . Hence, we can consider the set L/\sim , where the elements are the equivalence classes of \sim . In this way, $[x]$ denotes the equivalence class of $x \in L$, i.e the level which contains x . We also write $[L_i]$ in place of $[x_i]$ for any $x_i \in L_i$. Over L/\sim we can define also the following order relation:

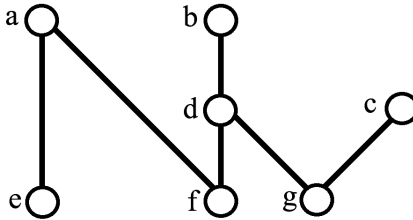
$$[x] \leq_{L/\sim} [y] \iff x \in L_i, y \in L_j \text{ and } j \leq i .$$

It is easy to prove that the above relation $\leq_{L/\sim}$ is effectively an order relation. In fact, $\leq_{L/\sim}$ defines a linear ordering over L/\sim . At this point, we are able to define the linearisation of a finite poset.

Definition 1. *Let (L, \leq) be a finite poset. The linear poset $(L/\sim, \leq_{L/\sim})$ defined above is called the linearisation of (L, \leq) .*

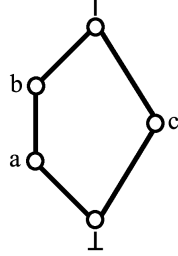
To the ease of representation, we denote by (\mathcal{L}, \preceq) the linearisation of (L, \leq) and have by definition $[L_i] \preceq [L_j]$ iff $j \leq i$. In the following, we illustrate some linearisation examples.

Example 1. Consider the poset represented by the following graph:



The levels of this poset are $L_0 = \{a, b, c\}$, $L_1 = \{d, e\}$, $L_2 = \{f, g\}$ and $L_i = \emptyset$ for $i \geq 3$. Then the linearisation of this poset is the chain $[f] \prec [d] \prec [a]$. □

Example 2. Consider the poset represented by the following graph:



Then the linearisation of this lattice is the chain $[\perp] \prec [a] \prec ([b] = [c]) \prec [\top]$. \square

Our linearisation satisfies the following properties.

Proposition 3. *Let (L, \leq) be a finite poset and let (\mathcal{L}, \preceq) be its linearisation. Then the following properties hold:*

1. *If (L, \leq) is a linear poset then $(\mathcal{L}, \preceq) \equiv (L, \leq)$.*
2. *If $x \leq y$ then $[x] \preceq [y]$. More specifically: if $x < y$ then $[x] \prec [y]$.*
3. *If $[x] \preceq [y]$, then for all $z \in [x]$ there exists $z' \in [y]$ such that $z \leq z'$. More specifically: if $[x] \prec [y]$, then for all $z \in [x]$ there exists $z' \in [y]$ such that $z < z'$.*

Proof. We begin with the proof of item (1). If \leq determines a linear ordering in L then (L, \leq) has a chain structure, i.e $(L, \leq) \equiv \perp < x_1 < \dots < x_n = \top$ for some $n \in \mathbb{N}$. It is easy to check that (\mathcal{L}, \preceq) is $[\perp] \prec [x_1] \prec \dots \prec [x_n] = [\top]$, with $[\perp] = \{\perp\}$, $[\top] = \{\top\}$ and $[x_i] = \{x_i\}$ for all $1 \leq i \leq n$. So, $(L, \leq) \equiv (\mathcal{L}, \preceq)$.

Item (2) is a direct consequence of Lemma 1.

Concerning item (3), suppose $[x] \prec [y]$ (the case $[x] = [y]$ holds trivially). Let L_i be the level associated with the class of $[x]$. Consider $z \in L_i$. Then, as $z \notin L_{i-1}$, there exists $z_1 \in L \setminus (L_0 \cup \dots \cup L_{i-2})$ such that $z < z_1$. Moreover, as $z \in L_i$ there does not exist $t \in L \setminus (L_0 \cup \dots \cup L_{i-1})$ such that $z < t$. In other words, z_1 belongs to L_{i-1} . If we continue with this procedure, we can build a chain as follows:

$$z < z_1 < \dots < z_j < \dots < z_i = \top$$

where each $z_j \in L_{i-j}$. To end the proof, we only have to select the element z_k in the chain such that $z_k \in [y]$. \square

Remark 1. Firstly we have just obtained the aim described in the introduction by taking the projection $p : L \rightarrow \mathcal{L}$ which assigns to each $x \in L$ the level $[x] \in \mathcal{L}$. Secondly, the item 3 implies that \preceq defines a H -ordering among the levels of a finite poset. \square

Proposition 4. *Let (L, \leq) be a finite poset, then the number of elements of (\mathcal{L}, \preceq) coincides with the maximum among the cardinalities of chains in (L, \leq) .*

Proof. As (L, \leq) is finite, every chain has a finite length. Let C be a chain $x_1 < \dots < x_n$ with maximal cardinality. Then, as the elements in the same level have to be incomparable, necessarily there exists at least n levels; i.e., \mathcal{L} has at least the same number of elements than the cardinality of C .

On the other hand, if the number of elements of \mathcal{L} is n , \mathcal{L} coincides with the chain $[x_{n-1}] \prec [x_{n-2}] \prec \dots \prec [x_1] \prec [x_0]$ for arbitrary $x_i \in L_i$.

Take an element $y_1 \in L_{n-1}$. By Proposition 3 item 3, there is an element $y_2 \in L_{n-2}$ such that $y_2 > y_1$. Similarly, by using the same result, there is an element $y_3 \in L_{n-3}$ such that $y_3 > y_2 > y_1$. By following this procedure we can build a chain $y_n > y_{n-1} > \dots > y_2 > y_1$ with cardinality that is then the number of elements of \mathcal{L} . Therefore the maximum among the cardinalities of chains of (L, \leq) is greater or equal than the number of elements of \mathcal{L} .

Unifying both inequalities, we conclude that the number of elements of (\mathcal{L}, \preceq) coincides with the maximum among the cardinalities of chains in (L, \leq) . \square

3.1 The Dual-linearisation

Clearly, the levels defined in the above section can be defined in a dual way. More exactly, we can define the dual-levels by taking incomparable elements in increasing order instead of decreasing order as in the procedure describe in the previous section. Therefore a dual linearisation procedure can be defined. In this section we will describe the dual-linearisation and we will provide results (or dual-results).

Let (L, \leq) be a poset, we define the dual-levels as follows:

$$\begin{aligned} L_0^* &= \{x \in L \mid \nexists y \in L \text{ such that } x > y\} \\ L_1^* &= \{x \in L \mid x \notin L_0^*, \nexists y \in L \setminus L_0^* \text{ such that } x > y\} \\ &\vdots \\ L_i^* &= \{x \in L \mid x \notin L_0^* \cup \dots \cup L_{i-1}^*, \\ &\quad \nexists y \in L / (L_0^* \cup \dots \cup L_{i-1}^*) \text{ such that } x > y\} \\ &\vdots \end{aligned}$$

Each L_i^* is called the i -dual-level of L . Lemma 1 is rewritten for dual-levels as follows:

Lemma 2. *Let (L, \leq) be a poset.*

1. *If $x \in L$ belongs to the level L_k then $x \notin L_i$ for all $i \neq k$.*
2. *If $x < y$, $x \in L_i$ and $y \in L_j$ then $i < j$.*
3. *If x and y belong to the same level then x and y are incomparable.*

Propositions 1 and 2 and Corollary 1 could be rewriting (and proved) substituting directly the levels L_i by the dual levels L_i^* . This feature allows us to define

the equivalence relation \sim_* as follows: $x \sim_* y$ if and only if x and y belong to the same dual-level. Hence, we can consider the quotient set L/\sim_* with the ordering:

$$[x] \preceq_* [y] \text{ if and only if } x \in L_i^*, y \in L_j^* \text{ and } i \leq j.$$

As for the order relation \preceq defined over \mathcal{L} , \preceq_* defines a linear ordering in L/\sim_* .

Definition 2. Let (L, \leq) be a finite poset. The linear poset $(L/\sim_*, \preceq_*)$ defined above is called the dual-linearisation of (L, \leq) .

We use the notation $(\mathcal{L}^*, \preceq_*)$ to denote the dual-linearisation of (L, \leq) , and have by definition $[L_i] \preceq_* [L_j]$ iff $i \leq j$, where we write $[L_i^*]$ in place of $[x_i]$ for any $x_i \in L_i^*$. Obviously, the dual-linearisation and the linearisation may not coincide.

Example 3. Consider the poset given in Example 2. Then dual-linearisation is the chain:

$$[\perp] \preceq_* ([c] = [a]) \preceq_* [b] \preceq_* [\top].$$

Note that in the linearisation $[a] \neq [c]$ whereas in the dual-linearisation $[a] = [c]$. \square

Proposition 3 can be rewriting for the dual-linearisation by changing slightly item 3.

Proposition 5. Let (L, \leq) be a finite poset and let $(\mathcal{L}^*, \preceq_*)$ be its dual-linearisation. Then the following properties hold:

1. If (L, \leq) is a linear poset then $(\mathcal{L}^*, \preceq_*) \equiv (L, \leq_L)$.
2. If $x \leq y$ then $[x] \preceq_* [y]$. More specifically: if $x < y$ then $[x] \prec_* [y]$.
3. If $[x] \preceq_* [y]$, then for all $z \in [y]$ there exists $z' \in [x]$ such that $z' \leq y$. More specifically: if $[x] \prec_* [y]$, then for all $z \in [y]$ there exists $z' \in [x]$ such that $z' < z$.

Remark 2. Note that item 3 in Proposition 5 shows the most important difference between the levels and the dual-levels. That is, the set of levels is H -ordered by \preceq whereas the set of dual-levels is S -ordered by \preceq_* . \square

As in the linearisation, the number of element in the dual linearisation is determined by the length of the maximal chains in (L, \leq) .

Proposition 6. Let (L, \leq) be a finite poset, then the number of elements of $(\mathcal{L}^*, \preceq_*)$ coincides with the maximum among the cardinalities of chains in (L, \leq) .

3.2 Relationships between linearisation and dual linearisation

The aim of this section is to determine a condition in (L, \leq) , which characterises the equivalence $(\mathcal{L}, \preceq) \equiv (\mathcal{L}^*, \preceq_*)$. Notice that if that equivalence holds then \preceq (or equivalently \preceq_*) determines an EM -ordering in the set of levels and in the set of dual-levels of (L, \leq) . It is important to note, as we show in Example 3, that usually (\mathcal{L}, \preceq) and $(\mathcal{L}^*, \preceq_*)$ are not equivalent. However, we can ensure that \mathcal{L} and \mathcal{L}^* have the same number of elements, thanks to Propositions 4 and 6.

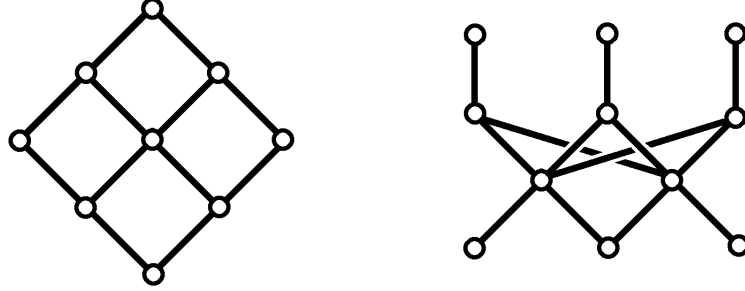
Corollary 2. *Let (L, \leq) be a finite poset. Then $|\mathcal{L}| = |\mathcal{L}^*|$.*

The condition which characterises the equivalence $(\mathcal{L}, \preceq) \equiv (\mathcal{L}^*, \preceq_*)$ is given in the following definition:

Definition 3. *A finite poset satisfies the equal length chain condition (ELCC) if and only if all maximal chains have the same length.*

Example 4. The poset given in example 2 does not satisfy the ELCC since $\perp < a < b < \top$ and $\perp < c < \top$ are two maximal chains with different length; 4 and 3 respectively. \square

Example 5. The following two posets satisfy the ELCC:



In the left poset every maximal chain has a length 5, whereas in the right poset every maximal chain has a length of 4. \square

Proposition 7. *Let (L, \leq) be a finite poset. (L, \leq) satisfies the ELCC if and only if $(\mathcal{L}, \preceq) \equiv (\mathcal{L}^*, \preceq_*)$; i.e for all $x \in L$:*

- *the equivalence class of x in \mathcal{L} and \mathcal{L}^* coincides;*
- *$[x] \preceq [y]$ if and only if $[x] \preceq_* [y]$.*

Proof. Let $n \in \mathbb{N}$ be the length of each maximal chain in (L, \leq) . Let us prove that each level L_i coincides with the dual level L_{n-i}^* for $i = 1, \dots, n$.

Consider $x \in L$ and suppose that $x \in L_i$ and $x \in \mathcal{L}_j^*$. By the Proposition 3 item 3, we can build the chain:

$$x < y_1 < y_2 < \dots < \top ,$$

where each $y_k \in L_{i-k}$.

Dually, by the Proposition 5 item 3, we can build the chain

$$\perp < \dots < z_2 < z_1 < x ,$$

where each $z_k \in L_{j-k}^*$. Unifying both chains we obtain the maximal chain

$$\perp < \dots < z_2 < z_1 < x < y_1 < y_2 < \dots < \top ,$$

whose length is $j + i$. As (L, \leq) satisfies the ELCC, $j + i = n$ and therefore $j = n - i$. Therefore $L_{n-i}^* = L_i$.

Finally, $[L_i] \preceq [L_j]$ iff $i \geq j$ iff $n - j \geq n - i$ iff $[L_{n-i}^*] \preceq_* [L_{n-j}^*]$ iff $[L_i] \preceq_* [L_j]$.

Let us prove the other implication. That is, $(\mathcal{L}, \preceq) \equiv (\mathcal{L}^*, \preceq_*)$ implies that (L, \leq) satisfies the ELCC.

We proceed by induction over the number of elements in L . For $i = 1$ the proof is trivial since every poset with only one point satisfies the ELCC and its linearisation and dual-linearisation coincide. Suppose that the number of elements of L is $n > 1$ and that the result is true for every poset with a number of elements less than n . Consider now $L \setminus L_0 = L \setminus \{x \in L : \nexists y \in L \text{ such that } x > y\}$. As $(\mathcal{L}, \preceq) \equiv (\mathcal{L}^*, \preceq_*)$ then necessarily $((L \setminus L_0), \leq) \equiv ((L \setminus L_0)^*, \preceq_*)$. Therefore, by induction hypothesis, $L \setminus L_0$ satisfies the ELCC. Let k be the length of every maximal chain in $L \setminus L_0$. Let us show that the length of every maximal chain in L is $k + 1$. Let C be a maximal chain of (L, \leq) with length k' . By Proposition 3 item 3, the maximal element in $z \in C$ has to belong to L_0 . Moreover $C \setminus \{z\}$ is a maximal chain of $L \setminus L_0$. Then by using the induction hypothesis, the length of $C \setminus \{z\}$ is k and therefore the length of C is $k + 1$. \square

4 Extending monotonic mappings over the linearisation

We take up again the issue to extend monotonic mappings over a linearized poset (\mathcal{L}, \preceq) or dually over $(\mathcal{L}^*, \preceq_*)$. The main problem of extending monotonic mappings in a linearisation procedure is that it *cannot* be done by homomorphic extension if the properties described in the introduction are satisfied. The following lemma shows this (unexpected) negative feature for a special family of posets: the lattices.⁴

Proposition 8. *Let (L_1, \leq_1) be a lattice such that \leq_1 is not a linear ordering. Then there does not exist a linear lattice (L_2, \leq_2) together with a map $p: L_1 \rightarrow L_2$ such that $\forall x, y \in L_1$:*

1. $x <_1 y$ implies $p(x) <_2 p(y)$;
2. for every monotonic mapping $f: L_1 \rightarrow L_1$, the mapping $\hat{f}: L_2 \rightarrow L_2$ defined by $\hat{f}(p(x)) = p(f(x))$ is well defined and monotonic.

Proof. As (L_1, \leq_1) is not a linear lattice, there exists two incomparable elements $a, b \in L_1$. Without loss of generality, let us suppose that $p(a) \leq_2 p(b)$. If $p(a) = p(b)$ then the extension in (L_2, \leq_2) of the monotonic mapping $f(x) = \sup\{x, a\}$ is not well defined: in fact

$$\begin{aligned} p(a) &= p(\sup\{a, a\}) = p(f(a)) \\ &= \hat{f}(p(a)) = \hat{f}(p(b)) \\ &= p(f(b)) = p(f(\sup\{b, a\})) \\ &>_2 p(a) . \end{aligned}$$

⁴ A lattice [3] is a poset (L, \leq) such that for every pair of elements $a, b \in L$ the operators supremum (\sup) and infimum (\inf) are defined.

If $p(a) <_2 p(b)$ then the extension in (L_2, \leq_2) of any monotonic mapping such that $f(a) = b$ and $f(b) = a$ cannot be monotonic. In fact, $p(a) <_2 p(b)$, but

$$\hat{f}(p(a)) = p(f(a)) = p(b) >_2 p(a) = p(f(b)) = \hat{f}(p(b)) .$$

□

However, on the positive side, if (L, \leq_L) and (M, \leq_M) are *finite* posets and $f: L^n \rightarrow M$ is a mapping, we can extend f over our linearisation $(\mathcal{L}, \preceq_{\mathcal{L}})$ and $(\mathcal{M}, \preceq_{\mathcal{M}})$, respectively, by the following two ways: $\hat{f}: \mathcal{L}^n \rightarrow \mathcal{M}$ and $\tilde{f}: \mathcal{L}^n \rightarrow \mathcal{M}$, where

$$\hat{f}([x_1], \dots, [x_n]) = \max\{[f(y_1, \dots, y_n)] \mid y_1 \in [x_1], \dots, y_n \in [x_n]\}$$

and

$$\tilde{f}([x_1], \dots, [x_n]) = \min\{[f(y_1, \dots, y_n)] \mid y_1 \in [x_1], \dots, y_n \in [x_n]\} .$$

Obviously \hat{f} and \tilde{f} are well-defined. Note the following abuse of notation in the above formulae: we use $[\cdot]$ to denote equivalence classes in $(\mathcal{L}, \preceq_{\mathcal{L}})$ as well as in $(\mathcal{M}, \preceq_{\mathcal{M}})$. We keep this abuse of notation in the rest of the section since it is clear when $[\cdot]$ is either an element of $(\mathcal{L}, \preceq_{\mathcal{L}})$ or an element of $(\mathcal{M}, \preceq_{\mathcal{M}})$.

More general mappings $f: L^n \rightarrow M^m$ can be extended to a mapping from \mathcal{L}^n to \mathcal{M}^m as well using the both ideas above. In this case f can be seen as m mappings $f_i: L^n \rightarrow M$ and \hat{f} and \tilde{f} are defined by:

$$\hat{f}([x_1], \dots, [x_n]) = (\hat{f}_1([x_1], \dots, [x_n]), \dots, \hat{f}_m([x_1], \dots, [x_n]))$$

$$\tilde{f}([x_1], \dots, [x_n]) = (\tilde{f}_1([x_1], \dots, [x_n]), \dots, \tilde{f}_m([x_1], \dots, [x_n])) .$$

The monotonicity (resp. antitonicity) of \hat{f} and \tilde{f} is determined by the monotony (resp. antitonicity) of each \hat{f}_i and \tilde{f}_i , respectively. As a consequence, we may restrict our study on mappings to the former case in which the codomain is a poset (M, \leq_M) . The following results holds:

Proposition 9. *Let (L, \leq_L) and (M, \leq_M) be two finite posets. Then, for every mapping $f: L^n \rightarrow M$, the following inequalities hold for all $x_1, \dots, x_n \in L$:*

$$\tilde{f}([x_1], \dots, [x_n]) \preceq_{\mathcal{M}} [f(x_1, \dots, x_n)] \preceq_{\mathcal{M}} \hat{f}([x_1], \dots, [x_n]) .$$

The following propositions show us that in the linearisation \hat{f} keeps the monotonicity of f whereas \tilde{f} keeps the antitonicity of f :

Proposition 10. *If $f: L^n \rightarrow M$ is monotonic then $\hat{f}: \mathcal{L}^n \rightarrow \mathcal{M}$ is monotonic as well.*

Proof. Assume $[x_i] \preceq_L [y_i]$ for $i = 1, \dots, n$. From Proposition 3 item 3, for any tuple $(v_1, \dots, v_n) \in [x_1] \times \dots \times [x_n]$ there exists a tuple $(t_1, \dots, t_n) \in [y_1] \times \dots \times [y_n]$

such that $v_i \leq t_i$ for all $i = 1, \dots, n$. Then, by using the monotonicity of f and Proposition 3 item 2 we have the inequality $[f(v_1, \dots, v_n)] \preceq_{\mathcal{M}} [f(t_1, \dots, t_n)]$. Therefore:

$$\begin{aligned} \hat{f}([x_1], \dots, [x_n]) &= \max\{[f(z_1, \dots, z_n)] \mid z_1 \in [x_1], \dots, z_n \in [x_n]\} \\ &\preceq_M \max\{[f(t_1, \dots, t_n)] \mid t_1 \in [y_1], \dots, t_n \in [y_n]\} \\ &\preceq_M \hat{f}([y_1], \dots, [y_n]) . \end{aligned}$$

□

In a similar way we may prove that:

Proposition 11. *If $f : L^n \rightarrow M$ is antitonic then $\tilde{f} : \mathcal{L}^n \rightarrow \mathcal{M}$ is antitonic as well.*

It is important to note that \hat{f} and \tilde{f} do not necessarily extend antitonic and monotonic mappings, respectively. The following example shows that feature:

Example 6. Over the poset given in Example 2 we consider the following monotonic mapping f and antitonic mapping g :

$$f(x) = \begin{cases} \perp & \text{if } x = \perp \\ c & \text{if } x = c \\ \top & \text{if } x = \top, a, b \end{cases} \quad g(x) = \begin{cases} \top & \text{if } x = \perp \\ c & \text{if } x = c \\ \perp & \text{if } x = \top, a, b \end{cases}$$

It is easy to check that, despite being f monotonic, the mapping

$$\tilde{f}([x]) = \begin{cases} [\perp] & \text{if } [x] = [\perp] \\ [c] & \text{if } [x] = [c] \\ \top & \text{if } [x] = [a] \end{cases}$$

is not monotonic ($[a] \preceq [c]$ and $\tilde{f}([a]) = [\top] \succ \tilde{f}([c]) = [c]$). On the other hand, it is also easy to check that, despite being g antitonic, the mapping :

$$\hat{g}([x]) = \begin{cases} [\top] & \text{if } [x] = [\perp] \\ [c] & \text{if } [x] = [c] \\ \perp & \text{if } [x] = [a] \end{cases}$$

is not antitonic ($[a] \leq [c]$ and $\hat{g}([a]) = [\perp] < \hat{g}([c]) = [c]$). □

Obviously, we can extend mappings through \hat{f} and \tilde{f} in the dual-linearisation as well, which are defined for $(\mathcal{L}^*, \preceq_*)$ similarly as for (\mathcal{L}, \preceq) . However, the behaviour of these extensions is dual. More precisely, the extension \hat{f} keeps the antitonicity, while \tilde{f} keeps the monotonicity in the dual-linearisation.

Proposition 12. *If $f : L^n \rightarrow M$ is antitonic then $\hat{f} : (\mathcal{L}^*)^n \rightarrow \mathcal{M}$ is antitonic as well.*

Proposition 13. *If $f : L^n \rightarrow L$ is monotonic then $\tilde{f} : (\mathcal{L}^*)^n \rightarrow \mathcal{M}$ is monotonic as well.*

The proofs of both results are essentially the same as for Proposition 10.

By using Proposition 12, Proposition 13 and the results provided in Section 3.2, we prove that:

Corollary 3. *Let (L, \leq_L) and (M, \leq_M) be two finite posets such that (L, \leq_L) satisfies the ELCC. Then if $f: L^n \rightarrow M$ is monotonic (resp. antitonic) then \hat{f} and \tilde{f} are monotonic (resp. antitonic).*⁵

5 Related Work & Conclusions

Linear extensions have usually be seen as the extension of a poset (L, \leq) to a linear poset (L, \leq') , where the set L is not changed [1, 2, 7, 8]. The main drawback of this kind of linearisation is that all of them prevents to extend monotonic mappings. This feature can be derived from Proposition 8, as the unique natural way to extend mappings to a linear extension is by homomorphic extension. Besides that, another drawback of linear extensions relates to the number of different possible linear extensions to look at. In [1], the authors resolved the problem of counting the number of linear extensions by showing that the problem is $\#P$ -complete, which may prevent it to be used in practice.

We instead have shown that it is possible to linearise posets, under the condition that they are finite, such that the monotonicity (resp. antitonicity) of arbitrary mappings is preserved.

References

1. G. Brightwell and P. Winkler. Counting linear extensions. In *STOC '91: Proceedings of the twenty-third annual ACM symposium on Theory of computing*, pages 175–181, 1991.
2. K. De Loof, B. De Baets, and H. De Meyer. Counting linear extension majority cycles in partially ordered sets on up to 13 elements. *Comput. Math. Appl.*, 59(4):1541–1547, 2010.
3. G. Grätzer. *General Lattice Theory*. Birkhäuser Verlag, Basel, 1978.
4. C. Li, K. C.-C. Chang, I. F. Ilyas, and S. Song. RankSQL: query algebra and optimization for relational top-k queries. In *Proceedings of the 2005 ACM SIGMOD International Conference on Management of Data (SIGMOD-05)*, pages 131–142, New York, NY, USA, 2005. ACM Press.
5. C. Li, K. Chen-chuan, I. F. Ilyas, C. Ihab, F. Ilyas, and S. Song. Ranksql: Query algebra and optimization for relational top-k queries. In *In SIGMOD*, pages 131–142. ACM Press, 2004.
6. T. Lukasiewicz and U. Straccia. Top-k retrieval in description logic programs under vagueness for the semantic web. In *In Proc. SUM-2007*, pages 16–30. Springer, 2007.
7. G. Patil and C. Taillie. Multiple indicators, partially ordered sets, and linear extensions: Multi-criterion ranking and prioritization. *Enviroment and Ecological Statistics*, 11:199–228, 2004.
8. G. Pruesse and F. Ruskey. Generating linear extensions fast. *SIAM J. Comput.*, 23(2):373–386, 1994.

⁵ Independently if \hat{f} and \tilde{f} is defined over either the linearisation or the dual-linearisation.